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Self-Duality in Nonlinear Electromagnetism * †

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Abstract

We discuss duality invariant interactions between electromagnetic fields and matter. The case of scalar fields is treated in some detail.

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Disclaimer

1 Duality rotations in four dimensions

The invariance of Maxwell's equations under "duality rotations" has been known for a long time. In relativistic notation these are rotations of the electromagnetic field strength $F_{\mu\nu}$ into its dual, which is defined by

$$\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\lambda\sigma}F^{\lambda\sigma}, \quad \tilde{\tilde{F}}_{\mu\nu} = -F_{\mu\nu}. \tag{1.1}$$

This invariance can be extended to electromagnetic fields in interaction with the gravitational field, which does not transform under duality. It is present in ungauged extended supergravity theories, in which case it generalizes to a nonabelian group [1]. In [2, 3] we studied the most general situation in which duality invariance of this type can occur. More recently [4] the duality invariance of the Born-Infeld theory, suitably coupled to the dilaton and axion [5], has been studied in considerable detail. In the present note we will show that most of the results of [4, 5] follow quite easily from our earlier general discussion. We shall also present some new results that were not made explicit in [2, 3], especially some properties of the scalar fields.

We begin by recalling and completing some basic results of our paper [2, 3]. Consider a Lagrangian which is a function of n real field strengths $F^a_{\mu\nu}$ and of some other fields χ^i and their derivatives $\chi^i_\mu = \partial_\mu \chi^i$:

$$L = L\left(F^a, \chi^i, \chi^i_\mu\right). \tag{1.2}$$

Since

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu, \tag{1.3}$$

we have the Bianchi identities

$$\partial^\mu \tilde{F}^a_{\mu\nu} = 0. \tag{1.4}$$

On the other hand, if we define

$$\tilde{G}^a_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\lambda\sigma}G^{a\lambda\sigma} \equiv 2\frac{\partial L}{\partial F^{\mu\nu}_a}, \tag{1.5}$$

we have the equations of motion

$$\partial^\mu \tilde{G}^a_{\mu\nu} = 0. \tag{1.6}$$

We consider an infinitesimal transformation of the form

$$\delta \begin{pmatrix} F \\ G \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} F \\ G \end{pmatrix}, \tag{1.7}$$

$$\delta\chi^i = \xi^i(\chi), \tag{1.8}$$

where A, B, C, D are real $n \times n$ constant infinitesimal matrices and $\xi^i(\chi)$ functions of the fields χ^i (but not of their derivatives), and ask under what circumstances the system of the equations of motion (1.4) and (1.6), as well as the equation of motion for the fields χ^i are invariant. The analysis of [2] shows that this is true if the matrices satisfy

$$A^T = -D, \quad B^T = B, \quad C^T = C, \tag{1.9}$$

(where the superscript T denotes the transposed matrix) and in addition the Lagrangian changes under (1.7) and (1.8) as

$$\delta L = \frac{1}{4}\left(FC\tilde{F} + GB\tilde{G}\right). \tag{1.10}$$

The relations (1.9) show that (1.7) is an infinitesimal transformation of the real noncompact symplectic group $Sp(2n, R)$ which has $U(n)$ as maximal compact subgroup. The finite form is

$$\begin{pmatrix} F' \\ G' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} F \\ G \end{pmatrix}, \tag{1.11}$$

where the $n \times n$ real submatrices satisfy

$$c^T a = a^T c, \quad b^T d = d^T b, \quad d^T a - b^T c = 1. \tag{1.12}$$

Notice that the Lagrangian is not invariant. In [2] we showed, however, that the derivative of the Lagrangian with respect to an invariant parameter *is* invariant. The invariant parameter could be a coupling constant or an external background field, such as the gravitational field, which does not change under duality rotations. It follows that the energy-momentum tensor, which can be obtained as the variational derivative of the Lagrangian with respect to the gravitational field, is invariant under duality rotations. No explicit check of its invariance, as was done in [4]–[7], is necessary.

The symplectic transformation (1.11) can be written in a complex basis as

$$\begin{pmatrix} F' + iG' \\ F' - iG' \end{pmatrix} = \begin{pmatrix} \phi_0 & \phi_1^* \\ \phi_1 & \phi_0^* \end{pmatrix} \begin{pmatrix} F + iG \\ F - iG \end{pmatrix}, \tag{1.13}$$

where $*$ means complex conjugation and the submatrices satisfy

$$\phi_0^T \phi_1 = \phi_1^T \phi_0, \quad \phi_0^\dagger \phi_0 - \phi_1^\dagger \phi_1 = 1. \tag{1.14}$$

The relation between the real and the complex basis is

$$\begin{aligned} 2a &= \phi_0 + \phi_0^* + \phi_1 + \phi_1^*, & -2ib &= \phi_0 - \phi_0^* + \phi_1 - \phi_1^*, \\ 2ic &= \phi_0 - \phi_0^* - \phi_1 + \phi_1^*, & 2d &= \phi_0 + \phi_0^* - \phi_1 - \phi_1^*. \end{aligned} \tag{1.15}$$

In [2, 3] we also described scalar fields valued in the quotient space $Sp(2n, R)/U(n)$. The quotient space can be parameterized by a complex symmetric $n \times n$ matrix $K = K^T$ whose real part has positive eigenvalues, or equivalently by a complex symmetric matrix $Z = Z^T$ such that $Z^\dagger Z$ has eigenvalues smaller than 1. They are related by

$$K = \frac{1 - Z^*}{1 + Z^*}, \quad Z = \frac{1 - K^*}{1 + K^*}. \tag{1.16}$$

These formulae are the generalization of the well-known map between the Lobachevskiĭ unit disk and the Poincaré upper half-plane: Z corresponds to the single complex variable parameterizing the unit disk; iK to the one parameterizing the upper half plane.

Under $Sp(2n, R)$

$$K \to K' = (-ic + dK)\,(a + ibK)^{-1}\,, \quad Z \to Z' = (\phi_1 + \phi_0^* Z)\,(\phi_0 + \phi_1^* Z)^{-1}\,, \tag{1.17}$$

or, infinitesimally,

$$\delta K = -iC + DK - KA - iKBK, \quad \delta Z = V + T^* Z - ZT - iZV^* Z, \tag{1.18}$$

where

$$T = -T^\dagger, \quad V = V^T. \tag{1.19}$$

The invariant nonlinear kinetic term for the scalar fields can be obtained from the Kähler metric [8]

$$\mathrm{Tr}\left(dK^* \frac{1}{K + K^*} dK \frac{1}{K + K^*}\right) = \mathrm{Tr}\left(dZ \frac{1}{1 - Z^* Z} dZ^* \frac{1}{1 - ZZ^*}\right) \tag{1.20}$$

which follows from the Kähler potential

$$\mathrm{Tr}\ln\left(1 - ZZ^*\right) \quad \text{or} \quad \mathrm{Tr}\ln(K + K^*), \tag{1.21}$$

which are equivalent up to a Kähler transformation. It is not hard to show that the metric (1.20) is positive definite. Throughout this paper we assume a flat background space-time metric; the generalization to a nonvanishing gravitational field is straightforward [2]–[5].

2 Born-Infeld theory

As a particularly simple example we consider the case when there is only one tensor $F_{\mu\nu}$ and no additional fields. Our equations become

$$\tilde{G} = 2\frac{\partial L}{\partial F}, \tag{2.1}$$

$$\delta F = \lambda G, \quad \delta G = -\lambda F \tag{2.2}$$

and

$$\delta L = \frac{1}{4}\lambda\left(G\tilde{G} - F\tilde{F}\right). \tag{2.3}$$

We have restricted the duality transformation to the compact subgroup $U(1) \cong SO(2)$, as appropriate when no additional fields are present. So $A = D = 0$, $B = -C = \lambda$.

Since L is a function of F alone, we can also write

$$\delta L = \delta F \frac{\partial L}{\partial F} = \lambda G \frac{1}{2}\tilde{G}. \tag{2.4}$$

Comparing (2.3) and (2.4), which must agree, we find

$$G\tilde{G} + F\tilde{F} = 0. \tag{2.5}$$

Together with (2.1), this is a partial differential equation for $L(F)$, which is the condition for the theory to be duality invariant. If we introduce the complex field

$$M = F - iG, \tag{2.6}$$

(2.5) can also be written as

$$M\widetilde{M}^* = 0. \tag{2.7}$$

Clearly, Maxwell's theory in vacuum satisfies (2.5), or (2.7), as expected. A more interesting example is the Born-Infeld theory [6], given by the Lagrangian

$$L = \frac{1}{g^2}\left(-\Delta^{\frac{1}{2}} + 1\right), \tag{2.8}$$

where

$$\Delta = -\det\left(\eta_{\mu\nu} + gF_{\mu\nu}\right) = 1 + \frac{1}{2}g^2F^2 - g^4\left(\frac{1}{4}F\tilde{F}\right)^2. \tag{2.9}$$

For small values of the coupling constant g (or for weak fields) L approaches the Maxwell Lagrangian. We shall use the abbreviation

$$\beta = \frac{1}{4}F\tilde{F}. \tag{2.10}$$

Then

$$\frac{\partial \Delta}{\partial F} = g^2 F - \beta g^4 \tilde{F}, \tag{2.11}$$

$$\tilde{G} = 2\frac{\partial L}{\partial F} = -\Delta^{-\frac{1}{2}} \left(F - \beta g^2 \tilde{F} \right), \tag{2.12}$$

and

$$G = \Delta^{-\frac{1}{2}} \left(\tilde{F} + \beta g^2 F \right). \tag{2.13}$$

Using (2.12) and (2.13), it is very easy to check that $G\tilde{G} = -F\tilde{F}$: the Born-Infeld theory is duality invariant. It is also not too difficult to check that $\partial L/\partial g^2$ is actually *invariant* under (2.2) and the same applies to $L - \frac{1}{4}F\tilde{G}$ (which in this case turns out to be equal to $-g^2 \partial L/\partial g^2$). These invariances are expected from our general theory.

It is natural to ask oneself whether the Born-Infeld theory is the most general physically acceptable solution of (2.5). This was investigated in [4] where a negative result was reached: more general Lagrangians satisfy (2.5), the arbitrariness depending on a function of one variable.

3 Schrödinger's formulation of Born's theory

Schrödinger [7] noticed that, for the Born-Infeld theory (2.8), F and G satisfy not only (2.5) [or (2.7)], but also the more restrictive relation

$$M \left(M\widetilde{M} \right) - \widetilde{M} M^2 = \frac{g^2}{8} \widetilde{M}^* \left(M\widetilde{M} \right)^2. \tag{3.1}$$

We have verified this by an explicit, although lengthy, calculation using (2.6), (2.12), (2.13) and (2.9). Schrödinger did not give the details of the calculation, presenting instead convincing arguments based on particular choices of reference systems. One can write (3.1) as

$$\frac{\partial \mathcal{L}}{\partial M} = g^2 \widetilde{M}^*, \tag{3.2}$$

where

$$\mathcal{L} = 4\frac{M^2}{\left(M\widetilde{M} \right)}, \tag{3.3}$$

and Schrödinger proposed $\mathcal{L}$ as the Lagrangian of the theory, instead of (2.8). Of course, $\mathcal{L}$ is a Lagrangian in a different sense than L, which is a field Lagrangian in the usual sense. Multiplying (3.1) by M and saturating the

unwritten indices $\mu\nu$, the left hand side vanishes, so that (2.7) follows. Using (3.1) it is easy to see that $\mathcal{L}$ is pure imaginary: $\mathcal{L} = -\mathcal{L}^*$. Schrödinger also pointed out that, if we introduce a map

$$\frac{1}{g^2}\frac{\partial \mathcal{L}}{\partial M} = f(M), \tag{3.4}$$

so that (3.1) or (3.2) can be written as

$$f(M) = \widetilde{M}^*, \tag{3.5}$$

the square of the map is the identity map

$$f\left(f(M)\right) = M. \tag{3.6}$$

This, together with the properties

$$f(\widetilde{M}) = -\tilde{f}(M), \quad f(M^*) = f(M)^*, \tag{3.7}$$

ensures the consistency of (3.1). Schrödinger used the Lagrangian (3.3) to construct a conserved, symmetric energy-momentum tensor. We have checked that, when suitably normalized, his energy-momentum tensor agrees with that of Born and Infeld up to an additive term proportional to $\eta_{\mu\nu}$.

Schrödinger's formulation is very clever and elegant and it has the advantage of being *manifestly* covariant under the duality rotation $M \to Me^{i\lambda}$ which is the finite form of (2.2). It is also likely that, as he seems to imply, his formulation is fully equivalent to the Born-Infeld theory (2.8), which would mean that the more restrictive equation (3.1) eliminates the remaining ambiguity in the solutions of (2.7). This virtue could actually be a weakness if one is looking for more general duality invariant theories.

4 Axion, dilaton and $SL(2,R)$

It is well known that, if there are additional scalar fields which transform nonlinearly, the compact group duality invariance can be enhanced to a duality invariance under a larger noncompact group (see, *e.g.*, [2] and references therein). In the case of the Born-Infeld theory, just as for Maxwell's theory, one complex scalar field suffices to enhance the $U(1) \cong SO(2)$ invariance to the $SU(1,1) \cong SL(2,R)$ noncompact duality invariance. This is pointed out

in [5], but it also follows the considerations of our paper [2]. We shall use the letter S instead of K for the scalar field, which, in the example under consideration, is a single complex field, not an $n \times n$ matrix. In today's more standard notation

$$S = S_1 - iS_2 = e^{-\phi} - ia, \quad S_1 > 0, \tag{4.1}$$

where ϕ is the dilaton and a is the axion. For $SL(2,R) \cong Sp(2,R)$, the matrices A, B, C, D are real numbers and $A = -D,\ B$ and C are independent. Then the infinitesimal $SL(2,R)$ transformation is

$$\delta S = -2AS - iBS^2 - iC. \tag{4.2}$$

For the $SO(2) \cong U(1)$ subgroup, $A = 0,\ B = -C = \lambda$,

$$\delta S = -i\lambda S^2 + i\lambda. \tag{4.3}$$

The scalar kinetic term, proportional to

$$\frac{\partial_\mu S^* \partial^\mu S}{(S + S^*)^2}, \tag{4.4}$$

is invariant under the nonlinear transformation (4.2) which, in terms of S_1, S_2, takes the form

$$\delta S_1 = -2AS_1 - iBS_1 S_2, \quad \delta S_2 = -2AS_2 + B\left(S_1^2 - S_2^2\right) + C. \tag{4.5}$$

The full noncompact duality transformation on $F_{\mu\nu}$ is now

$$\delta F = AF + BG, \quad \delta G = DF + DG, \quad D = -A, \tag{4.6}$$

and we are seeking a Lagrangian $\hat{L}(F, S)$ which satisfies

$$\delta \hat{L} = \frac{1}{4}\left(FC\tilde{F} + GB\tilde{G}\right), \tag{4.7}$$

where

$$\delta \hat{L} = \delta F \frac{\partial \hat{L}}{\partial F} + \delta S_1 \frac{\partial \hat{L}}{\partial S_1} + \delta S_2 \frac{\partial \hat{L}}{\partial S_2}, \tag{4.8}$$

and now

$$\tilde{G} = 2\frac{\partial \hat{L}}{\partial F}. \tag{4.9}$$

Equating (4.7) and (4.8) we see that $\hat{L}$ must satisfy

$$\frac{1}{4}\left(BG\tilde{G} - CF\tilde{F}\right) + \frac{1}{2}AF\tilde{G} + \delta S_1 \frac{\partial \hat{L}}{\partial S_1} + \delta S_2 \frac{\partial \hat{L}}{\partial S_2} = 0. \tag{4.10}$$

This equation can be solved as follows. Assume that $L(\mathcal{F})$ satisfies (2.1) and (2.5), *i.e.*

$$\mathcal{G}\tilde{\mathcal{G}} + \mathcal{F}\tilde{\mathcal{F}} = 0, \tag{4.11}$$

where

$$\tilde{\mathcal{G}} = 2\frac{\partial \mathcal{L}}{\partial \mathcal{F}}. \tag{4.12}$$

For instance, the Born-Infeld Lagrangian $L(\mathcal{F})$ does this. Then

$$\hat{L}(S, F) = L(S_1^{\frac{1}{2}} F) + \frac{1}{4} S_2 F \tilde{F} \tag{4.13}$$

satisfies (4.10). Indeed

$$\frac{\partial \hat{L}(S, F)}{\partial F} = \frac{\partial L}{\partial \mathcal{F}} S_1^{\frac{1}{2}} + \frac{1}{2} S_2 \tilde{F}. \tag{4.14}$$

So

$$\tilde{G} = \tilde{\mathcal{G}} S_1^{\frac{1}{2}} + S_2 \tilde{F}, \tag{4.15}$$

$$G = \mathcal{G} S_1^{\frac{1}{2}} + S_2 F, \tag{4.16}$$

where we have defined

$$\mathcal{F} = S_1^{\frac{1}{2}} F, \tag{4.17}$$

and $\tilde{\mathcal{G}}$ is given by (4.12). Now

$$G\tilde{G} = \mathcal{G}\tilde{\mathcal{G}} S_1 + S_2^2 F\tilde{F} + 2S_2 \mathcal{F}\tilde{\mathcal{G}}. \tag{4.18}$$

Using (4.11) in this equation we find

$$G\tilde{G} = \left(S_2^2 - S_1^2\right) F\tilde{F} + 2S_2 \mathcal{F}\tilde{\mathcal{G}}. \tag{4.19}$$

We also have

$$F\tilde{G} = \mathcal{F}\tilde{\mathcal{G}} + S_2 F\tilde{F}. \tag{4.20}$$

On the other hand, since

$$\frac{\partial L}{\partial S_1^{\frac{1}{2}}} = \frac{\partial \mathcal{L}}{\partial \mathcal{F}} F = \frac{1}{2} \tilde{\mathcal{G}} F, \tag{4.21}$$

we obtain

$$\frac{\partial \hat{L}}{\partial S_1} = \frac{\partial L}{\partial S_1^{\frac{1}{2}}} \frac{1}{2} S_1^{-\frac{1}{2}} = \frac{1}{4} \tilde{\mathcal{G}} S_1^{-\frac{1}{2}} F = \frac{1}{4} \tilde{\mathcal{G}} \mathcal{F} S_1^{-1}. \tag{4.22}$$

In addition

$$\frac{\partial \hat{L}}{\partial S_2} = \frac{1}{4} F \tilde{F}. \tag{4.23}$$

Using (4.19), (4.20), (4.22) and (4.23), together with (4.5), we see that (4.10) is satisfied. It is easy to check that the scale invariant combinations $\mathcal{F}$ and $\mathcal{G}$, given by (4.17) and (4.12) have the very simple transformation law

$$\delta \mathcal{F} = S_1 B \mathcal{G}, \quad \delta \mathcal{G} = -S_1 B \mathcal{F}, \tag{4.24}$$

i.e., they transform according to the $U(1) \cong SO(2)$ compact subgroup just as F and G in (2.2), but with the parameter λ replaced by $S_1 B$. If $L(\mathcal{F})$ is the Born-Infeld Lagrangian, the theory with scalar fields given by $\hat{L}$ in (4.13) can also be reformulated à la Schrödinger. From (4.16) and (4.17) solve for $\mathcal{F}$ and $\mathcal{G}$ in terms of F, G, S_1 and S_2. Then $\mathcal{M} = \mathcal{F} - i\mathcal{G}$ must satisfy the same equation (3.1) that M does when no scalar fields are present.

5 Connections to string theory

The duality rotations considered here are relevant to effective field theories from superstrings. The supersymmetric extension [9] of the Lagrangian (4.13) with $L(\mathcal{F}) = -\frac{1}{4}\mathcal{F}^2$ describes the dilaton plus Yang-Mills sector of effective $N = 1$ supergravity theories obtained from superstrings in the weak coupling ($S_1 \to \infty$) limit. The $SL(2, Z)$ subgroup of $SL(2, R)$ that is generated by the elements $4\pi S \to 1/4\pi S$ and $S \to S - i/4\pi$ relates different string theories [10] to one another. The generalization of [2] to two dimensional theories [11] has been used to derive the Kähler potential for moduli and matter fields in effective field theories from superstrings. In this case the scalars are valued on a coset space $\mathcal{K}/\mathcal{H}$, $\mathcal{K} \in SO(n, n)$, $\mathcal{H} \in SO(n) \times SO(n)$. The kinetic energy is invariant under $\mathcal{K}$, and the full classical theory is invariant under a subgroup of $\mathcal{K}$. String loop corrections reduces the invariance to a discrete subgroup that contains the $SL(2, Z)$ group generated by $T \to 1/T$, $T \to T - i$, where T is the squared radius of compactification in string units.

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